Varieties of general type with small volume

Chengxi Wang
UCLA

August 27, 2021

Volume

- (Hacon-M[c]Kernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
- For all smooth n-folds of general type, $vol(X)$ has a positive lower bound $a_n = 1/(r_n)^n$.

Volume

- (Hacon-McKernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
- For all smooth n-folds of general type, $vol(X)$ has a positive lower bound $a_n = 1/(r_n)^n$.

Volume

- (Hacon-McKernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
- For all smooth n-folds of general type, $vol(X)$ has a positive lower bound $a_n = 1/(r_n)^n$.

Volume

- (Hacon-M[c]Kernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
- For all smooth n-folds of general type, $vol(X)$ has a positive lower bound $a_n = 1/(r_n)^n$.

Volume

- (Hacon-M^cKernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
- For all smooth n-folds of general type, $vol(X)$ has a positive lower bound $a_n = 1/(r_n)^n$.

Volume

- (Hacon-M[c]Kernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
- For all smooth n-folds of general type, $vol(X)$ has a positive lower bound $a_n = 1/(r_n)^n$.

Volume

- (Hacon-McKernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
- For all smooth n-folds of general type, $vol(X)$ has a positive lower bound $a_n = 1/(r_n)^n$.

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1,1,2,5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4,5,6,7,23)$. $|mK_X|$ is birational $\Leftrightarrow$ $m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10,12,17,33,37,55)$, with volume $1/830280$ (by Brown and Kasprzyk).

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1,1,2,5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4,5,6,7,23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10,12,17,33,37,55)$, with volume $1/830280$ (by Brown and Kasprzyk).

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1,1,2,5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4,5,6,7,23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10,12,17,33,37,55)$, with volume $1/830280$ (by Brown and Kasprzyk).

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1, 1, 2, 5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4, 5, 6, 7, 23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10, 12, 17, 33, 37, 55)$, with volume $1/830280$ (by Brown and Kasprzyk).

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1,1,2,5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4,5,6,7,23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10,12,17,33,37,55)$, with volume $1/830280$ (by Brown and Kasprzyk).

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1, 1, 2, 5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4, 5, 6, 7, 23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10, 12, 17, 33, 37, 55)$, with volume $1/830280$ (by Brown and Kasprzyk).

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1,1,2,5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4,5,6,7,23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10,12,17,33,37,55)$, with volume $1/830280$ (by Brown and Kasprzyk).

Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1, 1, 2, 5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4, 5, 6, 7, 23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10, 12, 17, 33, 37, 55)$, with volume $1/830280$ (by Brown and Kasprzyk).

In high dimensions

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a smooth complex projective n-fold of general type with volume less than* $1/n^{(n \log n)/3}$.
2. $\exists$ *a smooth complex projective n-fold* X *of general type s.t. the linear system* $|mK_X|$ *does not give a birational embedding for any* $m \le n^{(\log n)/3}$.

Ballico, Pignatelli, and Tasin found smooth n-folds of general type with volume about $1/n^n$, and s.t. $|mK_X|$ does not give a birational embedding for m at most a constant times n^2.

In high dimensions

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a smooth complex projective n-fold of general type with volume less than* $1/n^{(n \log n)/3}$.
2. $\exists$ *a smooth complex projective n-fold* X *of general type s.t. the linear system* $|mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Ballico, Pignatelli, and Tasin found smooth n-folds of general type with volume about $1/n^n$, and s.t. $|mK_X|$ does not give a birational embedding for m at most a constant times n^2.

In high dimensions

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a smooth complex projective n-fold of general type with volume less than* $1/n^{(n\log n)/3}$.
2. $\exists$ *a smooth complex projective n-fold* X *of general type s.t. the linear system* $|mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Ballico, Pignatelli, and Tasin found smooth n-folds of general type with volume about $1/n^n$, and s.t. $|mK_X|$ does not give a birational embedding for m at most a constant times n^2.

In high dimensions

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a smooth complex projective n-fold of general type with volume less than* $1/n^{(n \log n)/3}$.
2. $\exists$ *a smooth complex projective n-fold* X *of general type s.t. the linear system* $|mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Ballico, Pignatelli, and Tasin found smooth n-folds of general type with volume about $1/n^n$, and s.t. $|mK_X|$ does not give a birational embedding for m at most a constant times n^2.

Noether-type inequality

- Surfaces of general type: $vol(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.
- (M. Chen and Z. Jiang) For every positive integer n, $\exists\, a_n > 0$, $b_n > 0$ s.t. $vol(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type.
- (J. Chen, M. Chen, and C. Jiang) 3-folds of general type: $vol(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$.(optimal constants)
- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large n.
  A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai

Noether-type inequality

- Surfaces of general type: $vol(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.
- (M. Chen and Z. Jiang) For every positive integer n, $\exists\ a_n > 0$, $b_n > 0$ s.t. $vol(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type.
- (J. Chen, M. Chen, and C. Jiang) 3-folds of general type: $vol(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$.(optimal constants)
- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large n.
  A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai

Noether-type inequality

- Surfaces of general type: $vol(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.
- (M. Chen and Z. Jiang) For every positive integer n, $\exists\ a_n > 0$, $b_n > 0$ s.t. $vol(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type.
- (J. Chen, M. Chen, and C. Jiang) 3-folds of general type: $vol(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$.(optimal constants)
- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large n.
  A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai

Noether-type inequality

- Surfaces of general type: $vol(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.
- (M. Chen and Z. Jiang) For every positive integer n, $\exists\ a_n > 0, b_n > 0$ s.t. $vol(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type.
- (J. Chen, M. Chen, and C. Jiang) 3-folds of general type: $vol(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$.(optimal constants)
- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large n.
  A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai

Noether-type inequality

- Surfaces of general type: $vol(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.
- (M. Chen and Z. Jiang) For every positive integer n, $\exists\ a_n > 0,\ b_n > 0$ s.t. $vol(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type.
- (J. Chen, M. Chen, and C. Jiang) 3-folds of general type: $vol(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$.(optimal constants)
- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large n.
  A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai

Noether-type inequality

- Surfaces of general type: $vol(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.
- (M. Chen and Z. Jiang) For every positive integer n, $\exists\ a_n > 0$, $b_n > 0$ s.t. $vol(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type.
- (J. Chen, M. Chen, and C. Jiang) 3-folds of general type: $vol(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$.(optimal constants)
- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large n.
  A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai

well-formed

- The weighted projective space $Y = P(a_0, \ldots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$, has trivial stabilizer group in codimension 1.)
- A general hypersurface of degree d is well-formed $\Leftrightarrow$ $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for each i.
- Reflexive sheaf $O(m)$ is a line bundle $\Leftrightarrow$ m is a multiple of every weight a_i.
- The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$.

well-formed

- The weighted projective space $Y = P(a_0, \ldots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$, has trivial stabilizer group in codimension 1.)
- A general hypersurface of degree d is well-formed $\Leftrightarrow$ $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for each i.
- Reflexive sheaf $O(m)$ is a line bundle $\Leftrightarrow$ m is a multiple of every weight a_i.
- The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$.

well-formed

- The weighted projective space $Y = P(a_0, \ldots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0} x_0, \ldots, t^{a_n} x_n)$, has trivial stabilizer group in codimension 1.)
- A general hypersurface of degree d is well-formed $\Leftrightarrow$ $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for each i.
- Reflexive sheaf $O(m)$ is a line bundle $\Leftrightarrow$ m is a multiple of every weight a_i.
- The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$.

well-formed

- The weighted projective space $Y = P(a_0, \ldots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$, has trivial stabilizer group in codimension 1.)
- A general hypersurface of degree d is well-formed $\Leftrightarrow$ $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for each i.
- Reflexive sheaf $O(m)$ is a line bundle $\Leftrightarrow$ m is a multiple of every weight a_i.
- The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$.

well-formed

- The weighted projective space $Y = P(a_0, \ldots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$, has trivial stabilizer group in codimension 1.)
- A general hypersurface of degree d is well-formed $\Leftrightarrow$ $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for each i.
- Reflexive sheaf $O(m)$ is a line bundle $\Leftrightarrow$ m is a multiple of every weight a_i.
- The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$.

well-formed

- The weighted projective space $Y = P(a_0, \dots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \dots, \widehat{a_j}, \dots, a_n) = 1$ for each j. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group G_m acts by $t(x_0, \dots, x_n) = (t^{a_0}x_0, \dots, t^{a_n}x_n)$, has trivial stabilizer group in codimension 1.)
- A general hypersurface of degree d is well-formed $\Leftrightarrow$ $\gcd(a_0, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \dots, \widehat{a_i}, \dots, a_n) = 1$ for each i.
- Reflexive sheaf $O(m)$ is a line bundle $\Leftrightarrow$ m is a multiple of every weight a_i.
- The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$.

Reid-Tai criterion for quotient singularities

For a positive integer r, let A^n/μ_r be the cyclic quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$ over a field, meaning that the group μ_r of rth roots of unity acts by $\zeta(x_1, \ldots, x_n) = (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n)$.

Assume that this description is well-formed in the sense that $\gcd(r, a_1, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for $j = 1, \ldots, n$. Then A^n/μ_r is canonical (resp. terminal) $\Leftrightarrow$

$$\sum_{j=1}^{n} (ia_j \text{ mod r}) \geq r$$

(resp. $> r$) for $i = 1, \ldots, r-1$.

Reid-Tai criterion for quotient singularities

For a positive integer r, let A^n/μ_r be the cyclic quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$ over a field, meaning that the group μ_r of rth roots of unity acts by $\zeta(x_1, \ldots, x_n) = (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n)$.

Assume that this description is well-formed in the sense that $\gcd(r, a_1, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for $j = 1, \ldots, n$. Then A^n/μ_r is canonical (resp. terminal) $\Leftrightarrow$

$$\sum_{j=1}^{n} (ia_j \bmod r) \geq r$$

(resp. $> r$) for $i = 1, \ldots, r-1$.

Reid-Tai criterion for quotient singularities

For a positive integer r, let A^n/μ_r be the cyclic quotient singularity of type $\frac{1}{r}(a_1, \dots, a_n)$ over a field, meaning that the group μ_r of rth roots of unity acts by $\zeta(x_1, \dots, x_n) = (\zeta^{a_1} x_1, \dots, \zeta^{a_n} x_n)$.

Assume that this description is well-formed in the sense that $\gcd(r, a_1, \dots, \widehat{a_j}, \dots, a_n) = 1$ for $j = 1, \dots, n$. Then A^n/μ_r is canonical (resp. terminal) $\Leftrightarrow$

$$\sum_{j=1}^{n} (ia_j \bmod r) \geq r$$

(resp. $> r$) for $i = 1, \dots, r-1$.

Reid-Tai criterion for quotient singularities

For a positive integer r, let A^n/μ_r be the cyclic quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$ over a field, meaning that the group μ_r of rth roots of unity acts by $\zeta(x_1, \ldots, x_n) = (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n)$.

Assume that this description is well-formed in the sense that $\gcd(r, a_1, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for $j = 1, \ldots, n$. Then A^n/μ_r is canonical (resp. terminal) $\Leftrightarrow$

$$\sum_{j=1}^{n} (ia_j \bmod \mathrm{r}) \geq r$$

(resp. $> r$) for $i = 1, \ldots, r-1$.

criterion for singularities of weighted projective spaces

It suffices for Y to be canonical or terminal at each coordinate point, $[0,\ldots,0,1,0,\ldots,0]$.

Lemma (Ballico, Pignatelli, and Tasin)

A well-formed weighted projective space $Y = P(a_0,\ldots,a_n)$ is canonical (resp. terminal) $\Leftrightarrow$ for each $0 \leq m \leq n$,

$$\sum_{j=0}^{n}(ia_j \bmod a_m) \geq a_m$$

(resp. $> a_m$) for $i = 1,\ldots,a_m - 1$.

criterion for singularities of weighted projective spaces

It suffices for Y to be canonical or terminal at each coordinate point, $[0, \ldots, 0, 1, 0, \ldots, 0]$.

Lemma (Ballico, Pignatelli, and Tasin)

A well-formed weighted projective space $Y = P(a_0, \ldots, a_n)$ is canonical (resp. terminal) $\Leftrightarrow$ for each $0 \leq m \leq n$,

$$\sum_{j=0}^{n} (ia_j \text{ mod } a_m) \geq a_m$$

(resp. $> a_m$) for $i = 1, \ldots, a_m - 1$.

Let $k \geq 2$ and $l \geq 0$ be integers. Ballico, Pignatelli, and Tasin consider hypersurface X of degree $d = (l+3)k(k+1)$ in weighted projective space $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Y is well-formed since $k \geq 2$.
- X is well-formed since d is a multiple of all weights.
- Y is canonical by Lemma. Check singularities of three types:

Let $k \geq 2$ and $l \geq 0$ be integers. Ballico, Pignatelli, and Tasin consider hypersurface X of degree $d = (l+3)k(k+1)$ in weighted projective space
$Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Y is well-formed since $k \geq 2$.
- X is well-formed since d is a multiple of all weights.
- Y is canonical by Lemma. Check singularities of three types:

Let $k \geq 2$ and $l \geq 0$ be integers. Ballico, Pignatelli, and Tasin consider hypersurface X of degree $d = (l+3)k(k+1)$ in weighted projective space
$Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Y is well-formed since $k \geq 2$.
- X is well-formed since d is a multiple of all weights.
- Y is canonical by Lemma. Check singularities of three types:

Let $k \geq 2$ and $l \geq 0$ be integers. Ballico, Pignatelli, and Tasin consider hypersurface X of degree $d = (l+3)k(k+1)$ in weighted projective space
$Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Y is well-formed since $k \geq 2$.
- X is well-formed since d is a multiple of all weights.
- Y is canonical by Lemma. Check singularities of three types:

Let $k \geq 2$ and $l \geq 0$ be integers. Ballico, Pignatelli, and Tasin consider hypersurface X of degree $d = (l+3)k(k+1)$ in weighted projective space $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Y is well-formed since $k \geq 2$.
- X is well-formed since d is a multiple of all weights.
- Y is canonical by Lemma. Check singularities of three types:

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)}),$
Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)}),$
Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)}),$Check
$(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
$ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$ $\begin{cases} (a)\ \text{Y is canonical.} \\ (b)\ \text{O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)},(k+1)^{(2k-1)},(k(k+1))^{(l)})$,
Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i=1,\dots,k-1$.
It's true since $i(k+1)=i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)},(k+1)^{(2k-2)},(k(k+1))^{(l)})$,
Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i=1,\dots,k$.
It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)},(k+1)^{(2k-1)},(k(k+1))^{(l-1)})$,Check
$(k+2)(ik \bmod k(k+1))+(2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i=1,\dots,k(k+1)-1$.
It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i=1,\dots,k(k+1)-1$,
and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
$ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

$$X \text{ is canonical} \Leftarrow \begin{cases} (a)\ \text{Y is canonical.} \\ (b)\ \text{O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check
   $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow \begin{cases} (a) \text{ Y is canonical.} \\ (b) \text{ O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check
   $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$ $\begin{cases} (a) \text{ Y is canonical.} \\ (b) \text{ O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
$ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

$$X \text{ is canonical} \Leftarrow \begin{cases} (a)\ \text{Y is canonical.} \\ (b)\ \text{O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$ $\begin{cases} (a) \text{ Y is canonical.} \\ (b) \text{ O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$, Check $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$, and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
$ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$ $\begin{cases} (a) \text{ Y is canonical.} \\ (b) \text{ O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)},(k+1)^{(2k-1)},(k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)},(k+1)^{(2k-2)},(k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)},(k+1)^{(2k-1)},(k(k+1))^{(l-1)})$,Check
   $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$ $\begin{cases} (a) \text{ Y is canonical.} \\ (b) \text{ O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check
   $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$ $\begin{cases} (a) \text{ Y is canonical.} \\ (b) \text{ O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

$$X \text{ is canonical} \Leftarrow \begin{cases} (a)\ \text{Y is canonical.} \\ (b)\ \text{O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check
   $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$ $\begin{cases} (a)\text{ Y is canonical.} \\ (b)\text{ O(d) is basepoint-free line bundle since} \\ \text{d>0 is a multiple of all the weights.} \end{cases}$

by Kollár's Bertini theorem.

1. $\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$,
   Check $(2k-1)(i(k+1) \bmod k) \geq k$ for $i = 1, \ldots, k-1$.
   It's true since $i(k+1) = i \geq 1 \bmod k$.
2. $\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)})$,
   Check $(k+2)(ik \bmod (k+1)) \geq k+1$ for $i = 1, \ldots, k$.
   It's true since $ik \geq 1 \bmod (k+1)$.
3. $\frac{1}{k(k+1)}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l-1)})$,Check
   $(k+2)(ik \bmod k(k+1)) + (2k-1)(i(k+1) \bmod k(k+1)) \geq k(k+1)$ for $i = 1, \ldots, k(k+1)-1$.
   It's true since $k \nmid i$ or $(k+1) \nmid i$ for $i = 1, \ldots, k(k+1)-1$,
   and $i(k+1) \geq k+1 \bmod k(k+1)$ if $k \nmid i$,
   $ik \geq k \bmod k(k+1)$ if $(k+1) \nmid i$.

X is canonical $\Leftarrow$
- (a) Y is canonical.
- (b) O(d) is basepoint-free line bundle since d>0 is a multiple of all the weights.

by Kollár's Bertini theorem.

quasi-smooth

- A closed subvariety X of a weighted projective space $P(a_0, \ldots, a_n)$ is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin.

Lemma (Iano-Fletcher)

A general hypersurface of degree d in $P(a_0, \ldots, a_n)$ is quasi-smooth $\Leftrightarrow$

- *either (1) $a_i = d$ for some i,*
- *or (2) for every nonempty subset I of $\{0, \ldots, n\}$,either (a) d is an N-linear combination of the numbers a_i with $i \in I$,or (b) there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an N-linear combination of the numbers a_i with $i \in I$.*

quasi-smooth

- A closed subvariety X of a weighted projective space $P(a_0, \ldots, a_n)$ is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin.

Lemma (Iano-Fletcher)

A general hypersurface of degree d in $P(a_0, \ldots, a_n)$ is quasi-smooth $\Leftrightarrow$

- *either (1) $a_i = d$ for some i,*
- *or (2) for every nonempty subset I of $\{0, \ldots, n\}$,either (a) d is an N-linear combination of the numbers a_i with $i \in I$,or (b) there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an N-linear combination of the numbers a_i with $i \in I$.*

quasi-smooth

- A closed subvariety X of a weighted projective space $P(a_0, \ldots, a_n)$ is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin.

Lemma (Iano-Fletcher)

A general hypersurface of degree d in $P(a_0, \ldots, a_n)$ is quasi-smooth $\Leftrightarrow$

- *either (1) $a_i = d$ for some i,*
- *or (2) for every nonempty subset I of $\{0, \ldots, n\}$,either (a) d is an N-linear combination of the numbers a_i with $i \in I$,or (b) there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an N-linear combination of the numbers a_i with $i \in I$.*

quasi-smooth

- A closed subvariety X of a weighted projective space $P(a_0, \ldots, a_n)$ is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin.

Lemma (Iano-Fletcher)

A general hypersurface of degree d in $P(a_0, \ldots, a_n)$ is quasi-smooth $\Leftrightarrow$

- *either (1) $a_i = d$ for some i,*
- *or (2) for every nonempty subset I of $\{0, \ldots, n\}$, either (a) d is an N-linear combination of the numbers a_i with $i \in I$, or (b) there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an N-linear combination of the numbers a_i with $i \in I$.*

quasi-smooth

- A closed subvariety X of a weighted projective space $P(a_0, \ldots, a_n)$ is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin.

Lemma (Iano-Fletcher)

A general hypersurface of degree d in $P(a_0, \ldots, a_n)$ is quasi-smooth $\Leftrightarrow$

- *either (1) $a_i = d$ for some i,*
- *or (2) for every nonempty subset I of $\{0, \ldots, n\}$,either (a) d is an N-linear combination of the numbers a_i with $i \in I$,or (b) there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an N-linear combination of the numbers a_i with $i \in I$.*

quasi-smooth

- A closed subvariety X of a weighted projective space $P(a_0, \ldots, a_n)$ is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin.

Lemma (Iano-Fletcher)

A general hypersurface of degree d in $P(a_0, \ldots, a_n)$ is quasi-smooth $\Leftrightarrow$

- *either (1) $a_i = d$ for some i,*
- *or (2) for every nonempty subset I of $\{0, \ldots, n\}$,either (a) d is an N-linear combination of the numbers a_i with $i \in I$,or (b) there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an N-linear combination of the numbers a_i with $i \in I$.*

compute the volume

A general hypersurface X of degree $d = (l+3)k(k+1)$ in $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed.} \\ (b) \text{ X is quasi-smooth since} \\ \text{d is a multiple of all the weights.} \end{cases}$$

  Thus $K_X = O_X(1)$ ample. So $vol(X) = K_X^n$, which is d divided by the product of all weights of Y.
  $vol(X) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}$.
- Let W be a resolution of singularities of X. W is a smooth complex projective n-fold of general type with $vol(W) = vol(X)$.

compute the volume

A general hypersurface X of degree $d = (l+3)k(k+1)$ in $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed.} \\ (b) \text{ X is quasi-smooth since} \\ \text{d is a multiple of all the weights.} \end{cases}$$

Thus $K_X = O_X(1)$ ample. So $vol(X) = K_X^n$, which is d divided by the product of all weights of Y.
$vol(X) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}$.

- Let W be a resolution of singularities of X. W is a smooth complex projective n-fold of general type with $vol(W) = vol(X)$.

compute the volume

A general hypersurface X of degree $d = (l+3)k(k+1)$ in $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed.} \\ (b) \text{ X is quasi-smooth since} \\ \text{d is a multiple of all the weights.} \end{cases}$$

Thus $K_X = O_X(1)$ ample. So $vol(X) = K_X^n$, which is d divided by the product of all weights of Y.

$vol(X) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}$.

- Let W be a resolution of singularities of X. W is a smooth complex projective n-fold of general type with $vol(W) = vol(X)$.

compute the volume

A general hypersurface X of degree $d = (l+3)k(k+1)$ in $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed.} \\ (b) \text{ X is quasi-smooth since} \\ \text{d is a multiple of all the weights.} \end{cases}$$

  Thus $K_X = O_X(1)$ ample. So $vol(X) = K_X^n$, which is d divided by the product of all weights of Y.
  $vol(X) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}$.

- Let W be a resolution of singularities of X. W is a smooth complex projective n-fold of general type with $vol(W) = vol(X)$.

compute the volume

A general hypersurface X of degree $d = (l+3)k(k+1)$ in $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed.} \\ (b) \text{ X is quasi-smooth since} \\ \text{d is a multiple of all the weights.} \end{cases}$$

  Thus $K_X = O_X(1)$ ample. So $vol(X) = K_X^n$, which is d divided by the product of all weights of Y.
  $vol(X) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}$.

- Let W be a resolution of singularities of X. W is a smooth complex projective n-fold of general type with $vol(W) = vol(X)$.

compute the volume

A general hypersurface X of degree $d = (l+3)k(k+1)$ in $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed.} \\ (b) \text{ X is quasi-smooth since} \\ \text{d is a multiple of all the weights.} \end{cases}$$

  Thus $K_X = O_X(1)$ ample. So $vol(X) = K_X^n$, which is d divided by the product of all weights of Y.
  $vol(X) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}$.

- Let W be a resolution of singularities of X. W is a smooth complex projective n-fold of general type with $vol(W) = vol(X)$.

Generalization

- Consider hypersurface X of degree $d = (6+l)k(k+1)(k+2)$ in $Y = P(1^{(3k+2)}, k^{(2k+2)}, (k+1)^{(2k+1)}, (k+2)^{(2k+2)}, (k(k+1))^{(2k+2)}, (k(k+2))^{(2k)}, ((k+1)(k+2))^{(2k-2)}, (k(k+1)(k+2))^{l})$, where $l \geq 0, k \geq 4$.
- Y is well-formed since 1 occurs more than once.
- X is well-formed and quasi-smooth since d is a multiple of all the weights.
- X is also canonical and $K_X = O_X(d - \sum a_i) = O_X(1)$.
- $vol(X) = \frac{(6+l)}{k^{6k+4+l-1}(k+1)^{6k+l}(k+2)^{6k+l-1}}$. This improves BPT's example.

Generalization

- Consider hypersurface X of degree $d = (6+l)k(k+1)(k+2)$ in $Y = P(1^{(3k+2)}, k^{(2k+2)}, (k+1)^{(2k+1)}, (k+2)^{(2k+2)}, (k(k+1))^{(2k+2)}, (k(k+2))^{(2k)}, ((k+1)(k+2))^{(2k-2)}, (k(k+1)(k+2))^{l})$, where $l \geq 0, k \geq 4$.
- Y is well-formed since 1 occurs more than once.
- X is well-formed and quasi-smooth since d is a multiple of all the weights.
- X is also canonical and $K_X = O_X(d - \sum a_i) = O_X(1)$.
- $vol(X) = \frac{(6+l)}{k^{6k+4+l-1}(k+1)^{6k+l}(k+2)^{6k+l-1}}$. This improves BPT's example.

Generalization

- Consider hypersurface X of degree $d = (6+l)k(k+1)(k+2)$ in $Y = P(1^{(3k+2)}, k^{(2k+2)}, (k+1)^{(2k+1)}, (k+2)^{(2k+2)}, (k(k+1))^{(2k+2)}, (k(k+2))^{(2k)}, ((k+1)(k+2))^{(2k-2)}, (k(k+1)(k+2))^{l})$, where $l \geq 0, k \geq 4$.
- Y is well-formed since 1 occurs more than once.
- X is well-formed and quasi-smooth since d is a multiple of all the weights.
- X is also canonical and $K_X = O_X(d - \sum a_i) = O_X(1)$.
- $vol(X) = \frac{(6+l)}{k^{6k+4+l-1}(k+1)^{6k+l}(k+2)^{6k+l-1}}$. This improves BPT's example.

Generalization

- Consider hypersurface X of degree $d = (6+l)k(k+1)(k+2)$ in $Y = P(1^{(3k+2)}, k^{(2k+2)}, (k+1)^{(2k+1)}, (k+2)^{(2k+2)}, (k(k+1))^{(2k+2)}, (k(k+2))^{(2k)}, ((k+1)(k+2))^{(2k-2)}, (k(k+1)(k+2))^{l})$, where $l \geq 0, k \geq 4$.
- Y is well-formed since 1 occurs more than once.
- X is well-formed and quasi-smooth since d is a multiple of all the weights.
- X is also canonical and $K_X = O_X(d - \sum a_i) = O_X(1)$.
- $vol(X) = \frac{(6+l)}{k^{6k+4+l-1}(k+1)^{6k+l}(k+2)^{6k+l-1}}$. This improves BPT's example.

Generalization

- Consider hypersurface X of degree $d = (6+l)k(k+1)(k+2)$ in $Y = P(1^{(3k+2)}, k^{(2k+2)}, (k+1)^{(2k+1)}, (k+2)^{(2k+2)}, (k(k+1))^{(2k+2)}, (k(k+2))^{(2k)}, ((k+1)(k+2))^{(2k-2)}, (k(k+1)(k+2))^{l})$, where $l \geq 0, k \geq 4$.
- Y is well-formed since 1 occurs more than once.
- X is well-formed and quasi-smooth since d is a multiple of all the weights.
- X is also canonical and $K_X = O_X(d - \sum a_i) = O_X(1)$.
- $vol(X) = \frac{(6+l)}{k^{6k+4+l-1}(k+1)^{6k+l}(k+2)^{6k+l-1}}$. This improves BPT's example.

Generalization

Let b, l, k be integers with $b \geq 2$, $l \geq 0$, and $k \geq 2b - 2$. For each subset I of $\{0, \ldots, b-1\}$, define (with j running through $0, 1, \ldots, b-1$):

$$k_I = \begin{cases} -1 + \sum_{j=0}^{b-1}(k+j) & \text{if } |I| = 0, \\ -|I| + \sum_{j \notin I}(k+j) & \text{if } 1 \leq |I| \leq b-2, \\ -(b-1) + 2\sum_{j \notin I}(k+j) & \text{if } |I| = b-1, \\ l & \text{if } |I| = b. \end{cases}$$

Generalization

Let b, l, k be integers with $b \geq 2$, $l \geq 0$, and $k \geq 2b - 2$. For each subset I of $\{0, \ldots, b-1\}$, define (with j running through $0, 1, \ldots, b-1$):

$$k_I = \begin{cases} -1 + \sum_{j=0}^{b-1}(k+j) & \text{if } |I| = 0, \\ -|I| + \sum_{j \notin I}(k+j) & \text{if } 1 \leq |I| \leq b-2, \\ -(b-1) + 2\sum_{j \notin I}(k+j) & \text{if } |I| = b-1, \\ l & \text{if } |I| = b. \end{cases}$$

Generalization

Let b, l, k be integers with $b \geq 2$, $l \geq 0$, and $k \geq 2b-2$. For each subset I of $\{0, \ldots, b-1\}$, define (with j running through $0, 1, \ldots, b-1$):

$$k_I = \begin{cases} -1 + \sum_{j=0}^{b-1}(k+j) & \text{if } |I| = 0, \\ -|I| + \sum_{j \notin I}(k+j) & \text{if } 1 \leq |I| \leq b-2, \\ -(b-1) + 2\sum_{j \notin I}(k+j) & \text{if } |I| = b-1, \\ l & \text{if } |I| = b. \end{cases}$$

Generalization

Let b, l, k be integers with $b \geq 2$, $l \geq 0$, and $k \geq 2b - 2$. For each subset I of $\{0, \ldots, b-1\}$, define (with j running through $0, 1, \ldots, b-1$):

$$k_I = \begin{cases} -1 + \sum_{j=0}^{b-1}(k+j) & \text{if } |I| = 0, \\ -|I| + \sum_{j \notin I}(k+j) & \text{if } 1 \leq |I| \leq b-2, \\ -(b-1) + 2\sum_{j \notin I}(k+j) & \text{if } |I| = b-1, \\ l & \text{if } |I| = b. \end{cases}$$

Generalization

Let b, l, k be integers with $b \geq 2$, $l \geq 0$, and $k \geq 2b - 2$. For each subset I of $\{0, \ldots, b-1\}$, define (with j running through $0, 1, \ldots, b-1$):

$$k_I = \begin{cases} -1 + \sum_{j=0}^{b-1}(k+j) & \text{if } |I| = 0, \\ -|I| + \sum_{j \notin I}(k+j) & \text{if } 1 \leq |I| \leq b-2, \\ -(b-1) + 2\sum_{j \notin I}(k+j) & \text{if } |I| = b-1, \\ l & \text{if } |I| = b. \end{cases}$$

- Let Y be the complex weighted projective space

$$P\Big(\big(\prod_{j\in I}(k+j)\big)^{(k_I)} : I \subset \{0,\ldots,b-1\} \Big).$$

Let $d = (2b+l)\prod_{j=0}^{b-1}(k+j)$. Then a general hypersurface X of degree d in Y has canonical singularities and $K_X = O_X(1)$.

- For X of sufficiently large dimension n, let $b = \lfloor (\log n)/(2\log 2) \rfloor$ and $k = \lfloor \sqrt{n}/(\log n)^2 \rfloor$. Then

$$vol(K_X) < 1/n^{(n\log n)/3}.$$

- Let Y be the complex weighted projective space

$$P\Big(\big(\prod_{j\in I}(k+j)\big)^{(k_I)} : I \subset \{0,\ldots,b-1\} \Big).$$

  Let $d = (2b+l)\prod_{j=0}^{b-1}(k+j)$. Then a general hypersurface X of degree d in Y has canonical singularities and $K_X = O_X(1)$.

- For X of sufficiently large dimension n, let $b = \lfloor (\log n)/(2\log 2) \rfloor$ and $k = \lfloor \sqrt{n}/(\log n)^2 \rfloor$. Then

$$vol(K_X) < 1/n^{(n\log n)/3}.$$

- Let Y be the complex weighted projective space

$$P\Big(\big(\prod_{j\in I}(k+j)\big)^{(k_I)} : I \subset \{0,\ldots,b-1\}\Big).$$

  Let $d = (2b+I)\prod_{j=0}^{b-1}(k+j)$. Then a general hypersurface X of degree d in Y has canonical singularities and $K_X = O_X(1)$.

- For X of sufficiently large dimension n, let $b = \lfloor(\log n)/(2\log 2)\rfloor$ and $k = \lfloor\sqrt{n}/(\log n)^2\rfloor$. Then

$$vol(K_X) < 1/n^{(n\log n)/3}.$$

- Let Y be the complex weighted projective space

$$P\left(\left(\prod_{j\in I}(k+j)\right)^{(k_I)} : I \subset \{0,\ldots,b-1\}\right).$$

  Let $d = (2b+l)\prod_{j=0}^{b-1}(k+j)$. Then a general hypersurface X of degree d in Y has canonical singularities and $K_X = O_X(1)$.

- For X of sufficiently large dimension n, let $b = \lfloor (\log n)/(2\log 2)\rfloor$ and $k = \lfloor \sqrt{n}/(\log n)^2 \rfloor$. Then

$$vol(K_X) < 1/n^{(n\log n)/3}.$$

Terminal Fano varieties.

- (Birkar) For each integer $n > 0$, $\exists$ a constant s_n s.t. for every terminal Fano n-fold X, $|-mK_X|$ gives a birational embedding for all $m \geq s_n$;
  and $\exists$ a constant $b_n > 0$ s.t. every terminal Fano n-fold X has $vol(-K_X) \geq b_n$.
- (J. Chen and M. Chen) The optimal cases:
  $dim = 2$, $X_6 \subset P(1,1,2,3)$ with volume 1,
  $dim = 3$, $X_{66} \subset P(1,5,6,22,33)$ with volume $1/330$,
- $dim = 4$, Brown-Kasprzyk's example
  $X_{3486} \subset P(1,41,42,498,1162,1743)$, with volume $1/498240036$

Terminal Fano varieties.

- (Birkar) For each integer $n > 0$, $\exists$ a constant s_n s.t. for every terminal Fano n-fold X, $|-mK_X|$ gives a birational embedding for all $m \geq s_n$;
  and $\exists$ a constant $b_n > 0$ s.t. every terminal Fano n-fold X has $vol(-K_X) \geq b_n$.
- (J. Chen and M. Chen) The optimal cases:
  $dim = 2$, $X_6 \subset P(1, 1, 2, 3)$ with volume 1,
  $dim = 3$, $X_{66} \subset P(1, 5, 6, 22, 33)$ with volume $1/330$,
- $dim = 4$, Brown-Kasprzyk's example $X_{3486} \subset P(1, 41, 42, 498, 1162, 1743)$, with volume 1/498240036

Terminal Fano varieties.

- (Birkar) For each integer $n > 0$, $\exists$ a constant s_n s.t. for every terminal Fano n-fold X, $|-mK_X|$ gives a birational embedding for all $m \geq s_n$;
  and $\exists$ a constant $b_n > 0$ s.t. every terminal Fano n-fold X has $vol(-K_X) \geq b_n$.
- (J. Chen and M. Chen) The optimal cases:
  $dim = 2$, $X_6 \subset P(1, 1, 2, 3)$ with volume 1,
  $dim = 3$, $X_{66} \subset P(1, 5, 6, 22, 33)$ with volume $1/330$,
- $dim = 4$, Brown-Kasprzyk's example $X_{3486} \subset P(1, 41, 42, 498, 1162, 1743)$, with volume $1/498240036$

Terminal Fano varieties.

- (Birkar) For each integer $n > 0$, $\exists$ a constant s_n s.t. for every terminal Fano n-fold X, $|-mK_X|$ gives a birational embedding for all $m \geq s_n$;
  and $\exists$ a constant $b_n > 0$ s.t. every terminal Fano n-fold X has $vol(-K_X) \geq b_n$.
- (J. Chen and M. Chen) The optimal cases:
  $dim = 2$, $X_6 \subset P(1,1,2,3)$ with volume 1,
  $dim = 3$, $X_{66} \subset P(1,5,6,22,33)$ with volume $1/330$,
- $dim = 4$, Brown-Kasprzyk's example $X_{3486} \subset P(1,41,42,498,1162,1743)$, with volume $1/498240036$

Terminal Fano varieties.

- (Birkar) For each integer $n > 0$, $\exists$ a constant s_n s.t. for every terminal Fano n-fold X, $|-mK_X|$ gives a birational embedding for all $m \geq s_n$;
  and $\exists$ a constant $b_n > 0$ s.t. every terminal Fano n-fold X has $vol(-K_X) \geq b_n$.
- (J. Chen and M. Chen) The optimal cases:
  $dim = 2$, $X_6 \subset P(1,1,2,3)$ with volume 1,
  $dim = 3$, $X_{66} \subset P(1,5,6,22,33)$ with volume $1/330$,
- $dim = 4$, Brown-Kasprzyk's example
  $X_{3486} \subset P(1,41,42,498,1162,1743)$, with volume
  1/498240036

Terminal Fano n-fold.

Adding two more weights equals to 1 in the weighted projective space Y.

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a complex terminal Fano n-fold X with* $vol(-K_X) < 1/n^{(n \log n)/3}$.
2. $\exists$ *a complex terminal Fano n-fold X s.t. the linear system* $|-mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Fujita's conjecture: for every smooth complex projective variety X of dimension n with an ample line bundle A, $K_X + (n+2)A$ is very ample.

Terminal Fano n-fold.

Adding two more weights equals to 1 in the weighted projective space Y.

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a complex terminal Fano n-fold* X *with* $vol(-K_X) < 1/n^{(n \log n)/3}$.
2. $\exists$ *a complex terminal Fano n-fold* X *s.t. the linear system* $|-mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Fujita's conjecture: for every smooth complex projective variety X of dimension n with an ample line bundle A, $K_X + (n+2)A$ is very ample.

Terminal Fano n-fold.

Adding two more weights equals to 1 in the weighted projective space Y.

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a complex terminal Fano n-fold X with* $vol(-K_X) < 1/n^{(n\log n)/3}$.
2. $\exists$ *a complex terminal Fano n-fold X s.t. the linear system* $|-mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Fujita's conjecture: for every smooth complex projective variety X of dimension n with an ample line bundle A, $K_X + (n+2)A$ is very ample.

Terminal Fano n-fold.

Adding two more weights equals to 1 in the weighted projective space Y.

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a complex terminal Fano n-fold X with* $vol(-K_X) < 1/n^{(n\log n)/3}$.
2. $\exists$ *a complex terminal Fano n-fold X s.t. the linear system* $|-mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Fujita's conjecture: for every smooth complex projective variety X of dimension n with an ample line bundle A, $K_X + (n+2)A$ is very ample.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-M^cKernan-Xu.
  $(X, \Delta) = \left(P^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1} \right)$, where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$. The volume of $K_X + \Delta$ is

$$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$

- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-M[c]Kernan-Xu.
  $(X, \Delta) = \left(P^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1}\right)$,
  where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$.
  The volume of $K_X + \Delta$ is
  $$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$
- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-M[c]Kernan-Xu.

$(X, \Delta) = \left(P^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1}\right)$,
where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$. The volume of $K_X + \Delta$ is

$$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$

- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-McKernan-Xu.

  $(X, \Delta) = \left(P^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1}\right),$

  where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$. The volume of $K_X + \Delta$ is

  $$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$

- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-McKernan-Xu.
  $(X, \Delta) = \left(P^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1}\right),$
  where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$. The volume of $K_X + \Delta$ is
  $$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$
- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-M[c]Kernan-Xu.

  $$(X, \Delta) = \left(P^n, \tfrac{1}{2}H_0 + \tfrac{2}{3}H_1 + \tfrac{6}{7}H_2 + \cdots + \tfrac{c_{n+1}-1}{c_{n+1}}H_{n+1} \right),$$

  where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$. The volume of $K_X + \Delta$ is

  $$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$

- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-M^cKernan-Xu.
  $(X, \Delta) = \left(P^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1}\right)$,
  where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$.
  The volume of $K_X + \Delta$ is
  $$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$
- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt pair

- Kollár proposed what may be the klt pair (X, Δ) of general type with standard coefficients that has minimum volume.
- There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-M^{c}Kernan-Xu.
  $(X, \Delta) = \left(P^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1}\right)$,
  where H_i are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence, $c_0 = 2$ and $c_{m+1} = c_m(c_m - 1) + 1$.
  The volume of $K_X + \Delta$ is

  $$1/(c_{n+2} - 1)^n < 1/2^{2^n}.$$

- The optimal example is "Hurwitz orbifold" of volume $1/42$ in dimension 1.

klt varieties

For a klt surface X with ample canonical class, the smallest known volume is $1/48983$, by an example of Alexeev and Liu.

In high dimensions:

Theorem (B. Totaro, C. Wang)

For every integer $n \geq 2$, $\exists$ a complex klt n-fold X with ample canonical class s.t. $vol(K_X) < 1/2^{2^n}$.

$\log(vol(K_X))$ of our klt varieties is asymptotic to $\log(vol(K_X + \Delta))$ in Kollár's klt pair above, as $n \to \infty$.

klt varieties

For a klt surface X with ample canonical class, the smallest known volume is $1/48983$, by an example of Alexeev and Liu. In high dimensions:

Theorem (B. Totaro, C. Wang)

For every integer $n \geq 2$, $\exists$ a complex klt n-fold X with ample canonical class s.t. $vol(K_X) < 1/2^{2^n}$.

$\log(vol(K_X))$ of our klt varieties is asymptotic to $\log(vol(K_X + \Delta))$ in Kollár's klt pair above, as $n \to \infty$.

klt varieties

For a klt surface X with ample canonical class, the smallest known volume is $1/48983$, by an example of Alexeev and Liu. In high dimensions:

Theorem (B. Totaro, C. Wang)

For every integer $n \geq 2$, $\exists$ a complex klt n-fold X with ample canonical class s.t. $vol(K_X) < 1/2^{2^n}$.

$\log(vol(K_X))$ of our klt varieties is asymptotic to $\log(vol(K_X + \Delta))$ in Kollár's klt pair above, as $n \to \infty$.

construct klt varieties with ample canonical class

- Construct weighted projective space $P(a_0, \ldots, a_{n+1})$.
- Sylvester's sequence: $c_0 = 2$, $c_1 = 3$, $c_2 = 7$, $c_3 = 43$, $c_4 = 1807, \ldots$ and $c_{n+1} = c_n(c_n - 1) + 1$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and

$$a_2 = y^3 + y + 1$$
$$a_1 = y(y+1)(1+a_2) - a_2$$
$$a_0 = y(1 + a_2 + a_1) - a_1.$$

- Let $x = 1 + a_0 + a_1 + a_2$, $d = yx = c_0 \cdots c_{n-2} x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$, and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2} x$ for $0 \leq i \leq n-2$.

construct klt varieties with ample canonical class

- Construct weighted projective space $P(a_0, \ldots, a_{n+1})$.
- Sylvester's sequence: $c_0 = 2$, $c_1 = 3$, $c_2 = 7$, $c_3 = 43$, $c_4 = 1807, \ldots$ and $c_{n+1} = c_n(c_n - 1) + 1$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and

$$\begin{aligned} a_2 &= y^3 + y + 1 \\ a_1 &= y(y+1)(1+a_2) - a_2 \\ a_0 &= y(1 + a_2 + a_1) - a_1. \end{aligned}$$

- Let $x = 1 + a_0 + a_1 + a_2$, $d = yx = c_0 \cdots c_{n-2}x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$, and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2}x$ for $0 \leq i \leq n-2$.

construct klt varieties with ample canonical class

- Construct weighted projective space $P(a_0, \ldots, a_{n+1})$.
- Sylvester's sequence: $c_0 = 2$, $c_1 = 3$, $c_2 = 7$, $c_3 = 43$, $c_4 = 1807, \ldots$ and $c_{n+1} = c_n(c_n - 1) + 1$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and

$$a_2 = y^3 + y + 1$$
$$a_1 = y(y+1)(1 + a_2) - a_2$$
$$a_0 = y(1 + a_2 + a_1) - a_1.$$

- Let $x = 1 + a_0 + a_1 + a_2$, $d = yx = c_0 \cdots c_{n-2} x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$, and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2} x$ for $0 \leq i \leq n-2$.

construct klt varieties with ample canonical class

- Construct weighted projective space $P(a_0, \ldots, a_{n+1})$.
- Sylvester's sequence: $c_0 = 2$, $c_1 = 3$, $c_2 = 7$, $c_3 = 43$, $c_4 = 1807, \ldots$ and $c_{n+1} = c_n(c_n - 1) + 1$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and

$$\begin{aligned} a_2 &= y^3 + y + 1 \\ a_1 &= y(y+1)(1+a_2) - a_2 \\ a_0 &= y(1 + a_2 + a_1) - a_1. \end{aligned}$$

- Let $x = 1 + a_0 + a_1 + a_2$,
  $d = yx = c_0 \cdots c_{n-2} x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$,
  and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2} x$ for $0 \leq i \leq n-2$.

construct klt varieties with ample canonical class

Let X be a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$.
Then X is a klt with dimension n and K_X ample,

$$vol(K_X) = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}.$$

Thus $vol(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.
It is about the $7/8$th power of the volume of Kollár's conjecturally optimal klt pair (X, Δ), since $vol(K_X + \Delta) = 1/(c_{n+2} - 1)^n \doteq 1/(c_{n-1} - 1)^{8n}$.

construct klt varieties with ample canonical class

Let X be a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$. Then X is a klt with dimension n and K_X ample,

$$vol(K_X) = \frac{1}{y^{n-3}x^{n-2}a_0a_1a_2}.$$

Thus $vol(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.
It is about the 7/8th power of the volume of Kollár's conjecturally optimal klt pair (X, Δ), since $vol(K_X + \Delta) = 1/(c_{n+2} - 1)^n \doteq 1/(c_{n-1} - 1)^{8n}$.

construct klt varieties with ample canonical class

Let X be a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$. Then X is a klt with dimension n and K_X ample,

$$vol(K_X) = \frac{1}{y^{n-3} x^{n-2} a_0 a_1 a_2}.$$

Thus $vol(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.
It is about the $7/8$th power of the volume of Kollár's conjecturally optimal klt pair (X, Δ), since $vol(K_X + \Delta) = 1/(c_{n+2} - 1)^n \doteq 1/(c_{n-1} - 1)^{8n}$.

construct klt varieties with ample canonical class

Let X be a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$. Then X is a klt with dimension n and K_X ample,

$$vol(K_X) = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}.$$

Thus $vol(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.
It is about the $7/8$th power of the volume of Kollár's conjecturally optimal klt pair (X, Δ), since $vol(K_X + \Delta) = 1/(c_{n+2} - 1)^n \doteq 1/(c_{n-1} - 1)^{8n}$.

construct klt varieties with ample canonical class

Let X be a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$. Then X is a klt with dimension n and K_X ample,

$$vol(K_X) = \frac{1}{y^{n-3}x^{n-2}a_0a_1a_2}.$$

Thus $vol(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.

It is about the 7/8th power of the volume of Kollár's conjecturally optimal klt pair (X, Δ), since $vol(K_X + \Delta) = 1/(c_{n+2} - 1)^n \doteq 1/(c_{n-1} - 1)^{8n}$.

construct klt varieties with ample canonical class

Let X be a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$. Then X is a klt with dimension n and K_X ample,

$$vol(K_X) = \frac{1}{y^{n-3}x^{n-2}a_0a_1a_2}.$$

Thus $vol(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.
It is about the $7/8$th power of the volume of Kollár's conjecturally optimal klt pair (X, Δ), since $vol(K_X + \Delta) = 1/(c_{n+2}-1)^n \doteq 1/(c_{n-1}-1)^{8n}$.

construct klt varieties with ample canonical class

Let X be a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$. Then X is a klt with dimension n and K_X ample,

$$vol(K_X) = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}.$$

Thus $vol(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.
It is about the $7/8$th power of the volume of Kollár's conjecturally optimal klt pair (X, Δ), since $vol(K_X + \Delta) = 1/(c_{n+2} - 1)^n \doteq 1/(c_{n-1} - 1)^{8n}$.

- some weights (the biggest ones) divide d and the ratios close to Sylvester's sequence c_i.
  Let $\frac{d}{a_{i+3}} \doteq c_i$ for $0 \leq i \leq n-2$. Let $d = c_0 \cdots c_{n-2}x$ for some integer x.
- $d - \sum a_i$ equals 1 $\Leftrightarrow x = 1 + a_0 + a_1 + a_2$.

- some weights (the biggest ones) divide d and the ratios close to Sylvester's sequence c_i.
Let $\frac{d}{a_{i+3}} \doteq c_i$ for $0 \le i \le n-2$. Let $d = c_0 \cdots c_{n-2} x$ for some integer x.
- $d - \sum a_i$ equals $1 \Leftrightarrow x = 1 + a_0 + a_1 + a_2$.

- some weights (the biggest ones) divide d and the ratios close to Sylvester's sequence c_i.
  Let $\frac{d}{a_{i+3}} \doteq c_i$ for $0 \leq i \leq n-2$. Let $d = c_0 \cdots c_{n-2} x$ for some integer x.
- $d - \sum a_i$ equals $1 \Leftrightarrow x = 1 + a_0 + a_1 + a_2$.

- some weights (the biggest ones) divide d and the ratios close to Sylvester's sequence c_i.
  Let $\frac{d}{a_{i+3}} \doteq c_i$ for $0 \leq i \leq n-2$. Let $d = c_0 \cdots c_{n-2} x$ for some integer x.
- $d - \sum a_i$ equals 1 $\Leftrightarrow x = 1 + a_0 + a_1 + a_2$.

- some weights (the biggest ones) divide d and the ratios close to Sylvester's sequence c_i.
  Let $\frac{d}{a_{i+3}} \doteq c_i$ for $0 \leq i \leq n-2$. Let $d = c_0 \cdots c_{n-2} x$ for some integer x.
- $d - \sum a_i$ equals $1 \Leftrightarrow x = 1 + a_0 + a_1 + a_2$.

- From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers d and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every i and there is a positive integer r such that:

1. $a_i | d$ if $i \geq r$,
2. $d - a_{r-1} \equiv 0 \pmod{a_{r-2}}, \ldots, d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_{r-1}}$.

- Choose other weights a_i to make X quasi-smooth. a_0, a_1, a_2 satisfy a "cycle" of congruences:

$$d - a_2 = 0(\bmod a_1), d - a_1 = 0(\bmod a_0), d - a_0 = 0(\bmod a_2),$$

- From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers d and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every i and there is a positive integer r such that:

1. $a_i | d$ if $i \geq r$,
2. $d - a_{r-1} \equiv 0 \pmod{a_{r-2}}, \ldots, d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_{r-1}}$.

- Choose other weights a_i to make X quasi-smooth. a_0, a_1, a_2 satisfy a "cycle" of congruences:

$$d-a_2 = 0(\bmod a_1), d-a_1 = 0(\bmod a_0), d-a_0 = 0(\bmod a_2),$$

- From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers d and $a_0, \dots, a_{n+1}$, a general hypersurface of degree d in $P(a_0, \dots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every i and there is a positive integer r such that:

1. $a_i | d$ if $i \geq r$,
2. $d - a_{r-1} \equiv 0 \pmod{a_{r-2}}, \dots, d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_{r-1}}$.

- Choose other weights a_i to make X quasi-smooth. a_0, a_1, a_2 satisfy a "cycle" of congruences:

$$d - a_2 = 0(\bmod a_1), d - a_1 = 0(\bmod a_0), d - a_0 = 0(\bmod a_2),$$

- From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers d and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every i and there is a positive integer r such that:

1. $a_i | d$ if $i \geq r$,
2. $d - a_{r-1} \equiv 0 \pmod{a_{r-2}}, \ldots, d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_{r-1}}$.

- Choose other weights a_i to make X quasi-smooth. a_0, a_1, a_2 satisfy a "cycle" of congruences:

$$d - a_2 = 0(\bmod a_1), d - a_1 = 0(\bmod a_0), d - a_0 = 0(\bmod a_2),$$

- From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers d and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every i and there is a positive integer r such that:

1. $a_i | d$ if $i \geq r$,
2. $d - a_{r-1} \equiv 0 \pmod{a_{r-2}}, \ldots, d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_{r-1}}$.

- Choose other weights a_i to make X quasi-smooth.

  a_0, a_1, a_2 satisfy a "cycle" of congruences:

$$d - a_2 = 0(\bmod a_1), d - a_1 = 0(\bmod a_0), d - a_0 = 0(\bmod a_2),$$

- From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers d and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every i and there is a positive integer r such that:

1. $a_i | d$ if $i \geq r$,
2. $d - a_{r-1} \equiv 0 \pmod{a_{r-2}}, \ldots, d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_{r-1}}$.

- Choose other weights a_i to make X quasi-smooth. a_0, a_1, a_2 satisfy a "cycle" of congruences:

$$d - a_2 = 0(\bmod a_1), d - a_1 = 0(\bmod a_0), d - a_0 = 0(\bmod a_2),$$

construct klt varieties with ample canonical class

- $dim = 2$, $X_{316} \subset P(158, 85, 61, 11)$ with volume $2/57035 \doteq 3.5 \times 10^{-5}$.
- $dim = 3$, $X_{340068} \subset P(170034, 113356, 47269, 9185, 223)$ with volume $1/5487505331993410 \doteq 1.8 \times 10^{-16}$.
- $dim = 4$, volume about 1.4×10^{-44}. The smallest known volume for a klt 4-fold with ample canonical class is about 1.4×10^{-47}.

construct klt varieties with ample canonical class

- $dim = 2$, $X_{316} \subset P(158, 85, 61, 11)$ with volume $2/57035 \doteq 3.5 \times 10^{-5}$.
- $dim = 3$, $X_{340068} \subset P(170034, 113356, 47269, 9185, 223)$ with volume $1/5487505331993410 \doteq 1.8 \times 10^{-16}$.
- $dim = 4$, volume about 1.4×10^{-44}. The smallest known volume for a klt 4-fold with ample canonical class is about 1.4×10^{-47}.

construct klt varieties with ample canonical class

- $dim = 2$, $X_{316} \subset P(158, 85, 61, 11)$ with volume $2/57035 \doteq 3.5 \times 10^{-5}$.
- $dim = 3$, $X_{340068} \subset P(170034, 113356, 47269, 9185, 223)$ with volume $1/5487505331993410 \doteq 1.8 \times 10^{-16}$.
- $dim = 4$, volume about 1.4×10^{-44}. The smallest known volume for a klt 4-fold with ample canonical class is about 1.4×10^{-47}.

construct klt varieties with ample canonical class

- $dim = 2$, $X_{316} \subset P(158, 85, 61, 11)$ with volume $2/57035 \doteq 3.5 \times 10^{-5}$.
- $dim = 3$, $X_{340068} \subset P(170034, 113356, 47269, 9185, 223)$ with volume $1/5487505331993410 \doteq 1.8 \times 10^{-16}$.
- $dim = 4$, volume about 1.4×10^{-44}. The smallest known volume for a klt 4-fold with ample canonical class is about 1.4×10^{-47}.

sketch of proof

Our construction of klt varieties with ample canonical class:

- Sylvester's sequence $\{c_i\}$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and
  $a_2 = y^3 + y + 1$,
  $a_1 = y(y+1)(1+a_2) - a_2$,
  $a_0 = y(1 + a_2 + a_1) - a_1$.
- Let $x = 1 + a_0 + a_1 + a_2$,
  $d = yx = c_0 \cdots c_{n-2}x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$,
  and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2}x$ for $0 \leq i \leq n-2$.
- $X \subset P(a_0, \ldots, a_{n+1})$ is a general hypersurface of degree d.

sketch of proof

Our construction of klt varieties with ample canonical class:

- Sylvester's sequence $\{c_i\}$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and
  $a_2 = y^3 + y + 1$,
  $a_1 = y(y+1)(1+a_2) - a_2$,
  $a_0 = y(1 + a_2 + a_1) - a_1$.
- Let $x = 1 + a_0 + a_1 + a_2$,
  $d = yx = c_0 \cdots c_{n-2} x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$,
  and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2} x$ for $0 \leq i \leq n-2$.
- $X \subset P(a_0, \ldots, a_{n+1})$ is a general hypersurface of degree d.

sketch of proof

Our construction of klt varieties with ample canonical class:

- Sylvester's sequence $\{c_i\}$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and
  $a_2 = y^3 + y + 1$,
  $a_1 = y(y+1)(1+a_2) - a_2$,
  $a_0 = y(1 + a_2 + a_1) - a_1$.
- Let $x = 1 + a_0 + a_1 + a_2$,
  $d = yx = c_0 \cdots c_{n-2}x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$,
  and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2}x$ for $0 \leq i \leq n-2$.
- $X \subset P(a_0, \ldots, a_{n+1})$ is a general hypersurface of degree d.

sketch of proof

Our construction of klt varieties with ample canonical class:

- Sylvester's sequence $\{c_i\}$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and
  $a_2 = y^3 + y + 1,$
  $a_1 = y(y+1)(1+a_2) - a_2,$
  $a_0 = y(1 + a_2 + a_1) - a_1.$
- Let $x = 1 + a_0 + a_1 + a_2$,
  $d = yx = c_0 \cdots c_{n-2}x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$,
  and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2}x$ for $0 \leq i \leq n-2$.
- $X \subset P(a_0, \ldots, a_{n+1})$ is a general hypersurface of degree d.

sketch of proof when $r = 3$

- X is klt since it has only cyclic quotient singularities.
- X is quasi-smooth since $d - a_2 = (y^2 + 1)a_1, d - a_1 = (y + 1)a_0, d - a_0 = (y^4 + 3y - 1)a_2$. (by Lemma)

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed} \\ (b) \text{ X is quasi-smooth} \end{cases}$$

- $vol(K_X) = vol(O_X(1)) = \frac{d}{a_0 \cdots a_{n+1}} = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}$

sketch of proof when $r = 3$

- X is klt since it has only cyclic quotient singularities.
- X is quasi-smooth since $d - a_2 = (y^2 + 1)a_1, d - a_1 = (y + 1)a_0, d - a_0 = (y^4 + 3y - 1)a_2$. (by Lemma)

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed} \\ (b) \text{ X is quasi-smooth} \end{cases}$$

- $vol(K_X) = vol(O_X(1)) = \frac{d}{a_0 \cdots a_{n+1}} = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}$

sketch of proof when $r = 3$

- X is klt since it has only cyclic quotient singularities.
- X is quasi-smooth since $d - a_2 = (y^2 + 1)a_1, d - a_1 = (y + 1)a_0, d - a_0 = (y^4 + 3y - 1)a_2$. (by Lemma)

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed} \\ (b) \text{ X is quasi-smooth} \end{cases}$$

- $vol(K_X) = vol(O_X(1)) = \frac{d}{a_0 \cdots a_{n+1}} = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}$

sketch of proof when $r = 3$

- X is klt since it has only cyclic quotient singularities.
- X is quasi-smooth since $d - a_2 = (y^2 + 1)a_1, d - a_1 = (y + 1)a_0, d - a_0 = (y^4 + 3y - 1)a_2$. (by Lemma)

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed} \\ (b) \text{ X is quasi-smooth} \end{cases}$$

- $vol(K_X) = vol(O_X(1)) = \frac{d}{a_0 \cdots a_{n+1}} = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}$

sketch of proof when $r = 3$

- In terms of $y = c_{n-1} - 1$, we have
  $a_2 = y^3 + y + 1 > y^3$,
  $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$,
  $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$,
  $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$.
  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

sketch of proof when $r = 3$

- In terms of $y = c_{n-1} - 1$, we have
  $a_2 = y^3 + y + 1 > y^3$,
  $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$,
  $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$,
  $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$.
  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

sketch of proof when $r = 3$

- In terms of $y = c_{n-1} - 1$, we have
  $a_2 = y^3 + y + 1 > y^3$,
  $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$,
  $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$,
  $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$.
  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

sketch of proof when $r = 3$

- In terms of $y = c_{n-1} - 1$, we have
  $a_2 = y^3 + y + 1 > y^3$,
  $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$,
  $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$,
  $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$.
  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

sketch of proof when $r = 3$

- In terms of $y = c_{n-1} - 1$, we have
  $a_2 = y^3 + y + 1 > y^3$,
  $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$,
  $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$,
  $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$.
  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

sketch of proof when $r = 3$

- In terms of $y = c_{n-1} - 1$, we have
  $a_2 = y^3 + y + 1 > y^3$,
  $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$,
  $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$,
  $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$.
  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

sketch of proof when $r = 3$

- In terms of $y = c_{n-1} - 1$, we have
  $a_2 = y^3 + y + 1 > y^3$,
  $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$,
  $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$,
  $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$.
  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

Better klt varieties with ample canonical class

For any odd number $r \geq 3$ and any dimension $n \geq r-1$, we give an example with weights chosen to satisfy a cycle of r congruences.

- $\frac{\log(vol(K_X))}{\log(vol(K_Y+\Delta))} \to \frac{2^r-1}{2^r}$ as $n \to \infty$.
- For $r = 3$, this is the example above.
- When $r = 5$, $n = 4$, it is a a general hypersurface of degree 147565206676 in $P(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361)$ with $\doteq 7.4 \times 10^{-45}$. (Better)

Better klt varieties with ample canonical class

For any odd number $r \geq 3$ and any dimension $n \geq r - 1$, we give an example with weights chosen to satisfy a cycle of r congruences.

- $\frac{\log(vol(K_X))}{\log(vol(K_Y+\Delta))} \to \frac{2^r-1}{2^r}$ as $n \to \infty$.
- For $r = 3$, this is the example above.
- When $r = 5$, $n = 4$, it is a a general hypersurface of degree 147565206676 in $P(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361)$ with $\doteq 7.4 \times 10^{-45}$. (Better)

Better klt varieties with ample canonical class

For any odd number $r \geq 3$ and any dimension $n \geq r-1$, we give an example with weights chosen to satisfy a cycle of r congruences.

- $\frac{\log(vol(K_X))}{\log(vol(K_Y+\Delta))} \to \frac{2^r-1}{2^r}$ as $n \to \infty$.
- For $r = 3$, this is the example above.
- When $r = 5$, $n = 4$, it is a a general hypersurface of degree 147565206676 in $P(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361)$ with $\doteq 7.4 \times 10^{-45}$. (Better)

Better klt varieties with ample canonical class

For any odd number $r \geq 3$ and any dimension $n \geq r - 1$, we give an example with weights chosen to satisfy a cycle of r congruences.

- $\frac{\log(vol(K_X))}{\log(vol(K_Y+\Delta))} \to \frac{2^r-1}{2^r}$ as $n \to \infty$.
- For $r = 3$, this is the example above.
- When $r = 5$, $n = 4$, it is a a general hypersurface of degree 147565206676 in $P(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361)$ with $\doteq 7.4 \times 10^{-45}$. (Better)

Better klt varieties with ample canonical class

For any odd number $r \geq 3$ and any dimension $n \geq r-1$, we give an example with weights chosen to satisfy a cycle of r congruences.

- $\frac{\log(vol(K_X))}{\log(vol(K_Y+\Delta))} \to \frac{2^r-1}{2^r}$ as $n \to \infty$.
- For $r = 3$, this is the example above.
- When $r = 5$, $n = 4$, it is a a general hypersurface of degree 147565206676 in $P(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361)$ with $\doteq 7.4 \times 10^{-45}$. (Better)

Better klt varieties with ample canonical class

For any odd number $r \geq 3$ and any dimension $n \geq r-1$, we give an example with weights chosen to satisfy a cycle of r congruences.

- $\frac{\log(vol(K_X))}{\log(vol(K_Y+\Delta))} \to \frac{2^r-1}{2^r}$ as $n \to \infty$.
- For $r = 3$, this is the example above.
- When $r = 5$, $n = 4$, it is a a general hypersurface of degree 147565206676 in $P(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361)$ with $\doteq 7.4 \times 10^{-45}$. (Better)

Thank you!